

PREPARED FOR SUBMISSION TO JHEP

UTTG-16-16

Noether Charge, Black Hole Volume, and Complexity

Josiah Couch,^a Willy Fischler,^a and Phuc H. Nguyen^a

^a*Theory Group, Department of Physics and Texas Cosmology Center, University of Texas
Austin, TX 78712, USA*

E-mail: josiah.couch@utexas.edu, fischler@physics.utexas.edu,
phn229@physics.utexas.edu

ABSTRACT: In this paper, we study the physical significance of the thermodynamic volumes of AdS black holes using the Noether charge formalism of Iyer and Wald. After applying this formalism to study the extended thermodynamics of an R-charged black hole, we discuss how extended thermodynamics interacts with the recent complexity = action proposal of Brown et al. (CA-duality). We, in particular, discover that their proposal for the late time rate of change of complexity has a nice decomposition in terms of thermodynamic quantities reminiscent of the Smarr relation. This decomposition strongly suggests a geometric, and via CA-duality holographic, interpretation for the thermodynamic volume of an AdS black hole. We go on to discuss the role of thermodynamics in complexity = action for a number of black hole solutions, and then point out the possibility of an alternate proposal, which we dub “complexity = volume 2.0”. In this alternate proposal the complexity would be thought of as the spacetime volume of the Wheeler-DeWitt patch, which we believe could resolve some of the issues with CA-duality in the case the AdS-RN black hole.

Contents

1	Introduction	1
2	Volume and Iyer-Wald Formalism	2
2.1	Iyer-Wald	3
2.2	Application to the R-charged Black Hole	4
3	Volume and Complexity: Schwarzschild-AdS	7
3.1	Review of Brown et al.	8
3.2	Volume as Complexity Growth	9
3.3	Smarr Relation as Complexity Growth	10
3.4	Complexity = Volume 2.0	11
4	Volume and Complexity: Conserved Charges	12
4.1	Charged Black Hole	13
4.2	Rotating Black Hole	15
5	Discussion and conclusion	17
6	Acknowledgement	19

1 Introduction

The laws of black hole thermodynamics, at least in their traditional formulation [1–4], do not include a pressure-volume conjugate pair. This conspicuous absence is perhaps related to the difficulty of defining the volume of a black hole in a coordinate-invariant way: unlike the area of the horizon, a naïve integration over the interior of a black hole depends on the foliation of spacetime. A number of relativists [5–9], and more recently high energy physicists [10], have suggested that the pressure should be identified as the cosmological constant. In this framework, dubbed the extended black hole thermodynamics or “black hole chemistry”, the ADM mass of the black hole is reinterpreted as the enthalpy H of the system rather than its internal energy U . The volume, then, can be defined in the usual thermodynamical way to be:

$$V = \left(\frac{\partial H}{\partial P} \right)_S \quad (1.1)$$

Fascinatingly enough, in simple cases such as the AdS-Schwarzschild or AdS-Reissner-Nordstrom (AdS-RN) black hole, the thermodynamic volume coincides with a naïve integration over the “black hole interior”:

$$V = \frac{4}{3}\pi r_+^3 = \int_0^{r_+} \sqrt{-g} dr d\Omega_2^2 \quad (1.2)$$

In more complicated cases, such as rotating holes or solutions with hair, the thermodynamical volume is less intuitive, and it is an interesting question to ask how the volume arises as an integral of some local quantity over some region of spacetime, in a way similar to (1.2). For a selection of work on or related to this topic, we refer to [11–17].

In the first half of the paper, we present a systematic way to do this via the Noether charge formalism (or Iyer-Wald formalism). A powerful way to derive the first law of black hole thermodynamics, the Iyer-Wald formalism has yielded deep insights into the nature of black hole entropy, so it is a natural step to extend the formalism to derive black hole volumes.

In recent years, the Iyer-Wald formalism has proved useful as a means to translate between the geometry in the bulk to quantum information theoretic quantities on the boundary, starting with [18] where the Iyer-Wald formalism was used to derive the linearized equation of motion in the bulk from the first law of entanglement on the boundary in pure AdS. To give a few examples, the Iyer-Wald formalism was used in [19] to relate matter in the bulk to the relative entropy on the boundary, in [20] to relate canonical energy in the bulk to the quantum Fisher information on the boundary, and in [21] to relate quantum information inequalities to gravitational positive energy theorems.

We also note here that various notions of volume in the bulk have been identified with certain quantum information theoretic notions such as the complexity [22–25] or the fidelity [26]. In the light of all these papers, it is very suggestive that the thermodynamical volume also admits a quantum information theoretic interpretation, and we will find in this paper that it is indeed so. In the second half of the paper, we will relate the thermodynamic volume (and also the Smarr relation) to the complexity as per the proposals in [27, 28].

The rest of the paper is organized as follows: In Section 2, we briefly review the Iyer-Wald formalism (with varying cosmological constant) and apply it to the R-charged black hole. In Section 3, we move on to discuss the connection between the extended thermodynamics and the complexity in the simple case of the AdS-Schwarzschild black hole. In Section 4, we extend this connection with the complexity to a black hole with conserved charges (i.e. electrically charged and rotating holes). In Section 5, we conclude and discuss future work.

2 Volume and Iyer-Wald Formalism

The main goal of this section is to derive the volume using the Iyer-Wald formalism, more specifically a variation of this formalism developed in [13] in the context of holographic entanglement entropy, which allows for varying couplings in the action. First, we will briefly summarize the formalism used in [13] and contrast it with the usual Iyer-Wald formalism. Then we will apply it to compute the volume of a hairy solution: the R-charged black hole in 4 dimensions.

2.1 Iyer-Wald

Here we will review the Iyer-Wald formalism [29, 30]. The formalism requires a diffeomorphism invariant action $S = \int L$ together with a bifurcate timelike Killing vector field ξ ¹. Consider some general variation of the Lagrangian. For an action including a cosmological constant which is allowed to vary, this may be written as

$$\delta L = d\Theta + \sum_{\phi} E^{\phi} \delta\phi + \frac{\partial L}{\partial \Lambda} \delta\Lambda \quad (2.1)$$

Where E^{ϕ} is the equation of motion form for the field ϕ , and where the sum over ϕ runs over the entire field content of the theory. In the case where this variation is due to applying a diffeomorphism generated by a vector field ζ , this becomes

$$\delta_{\zeta} L = d\Theta(\delta_{\zeta}\phi) + \sum_{\phi} E^{\phi} \delta_{\zeta}\phi \quad (2.2)$$

In this case, since our action is diffeomorphism invariant, we may apply Noether's theorem to derive

$$J(\zeta) = \Theta(\delta_{\zeta}\phi) - \zeta \cdot L \quad (2.3)$$

Note that by Cartan's magic formula $\delta L = d(\zeta \cdot L)$, and so this is just the usual Noether current. We may then find a Noether charge form Q such that

$$J(\zeta) = dQ(\zeta) \quad (2.4)$$

We would then like to construct a form

$$\chi = \delta Q(\xi) - \xi \cdot \Theta(\delta\phi) \quad (2.5)$$

where $\delta\phi$ is some arbitrary variation of the fields, $Q(\xi)$ is the charge form of the killing vector ξ , and $\delta Q(\xi)$ is the variation of $Q(\xi)$ due to $\delta\phi$. Now we have by the magic formula that

$$d(\xi \cdot \Theta(\delta\phi)) = \delta_{\xi} \Theta(\delta\phi) - \xi \cdot d\Theta(\delta\phi) \quad (2.6)$$

and that on-shell

$$\xi \cdot d\Theta(\delta\phi) = \xi \cdot \delta L(\delta\phi) - \xi \cdot \frac{\partial L}{\partial \Lambda} \delta\Lambda = \delta(\xi \cdot L) - \xi \cdot \frac{\partial L}{\partial \Lambda} \delta\Lambda \quad (2.7)$$

On the other hand

$$\xi \cdot L = \Theta(\delta_{\xi}\phi) - J(\xi) \quad (2.8)$$

¹The formalism contains ambiguities which are discussed in [31].

Putting this together we get

$$d\chi = \delta(dQ - J) + \delta\Theta(\delta_\xi\phi) - \delta_\xi\Theta(\delta\phi) - \xi \cdot \frac{\partial L}{\partial \Lambda} \delta\Lambda \quad (2.9)$$

Because ξ is a killing vector $\delta\Theta(\delta_\xi\phi) - \delta_\xi\Theta(\delta\phi)$ vanishes. We therefore get on-shell

$$d\chi = -\xi \cdot \frac{\partial L}{\partial \Lambda} \delta\Lambda \quad (2.10)$$

Applying Stokes' theorem to this form on a region Σ of a constant time slice bounded by the bifurcate killing horizon and the conformal boundary at infinity then yields

$$\int_\Sigma d\chi = -\delta\Lambda \int_\Sigma \xi \cdot \frac{\partial L}{\partial \Lambda} = \int_\infty \chi - \int_{\mathcal{H}} \chi \quad (2.11)$$

In the case of a black hole spacetime, this reduces to the extended first law of black hole thermodynamics upon evaluation of the integrals, where roughly speaking, the second term from the left above gives rise to the VdP . For more details of the derivation of (2.11), we will refer to [13].

2.2 Application to the R-charged Black Hole

In this subsection, we illustrate the formalism above to study the volume of the R-charged black hole in 4 dimensions. We have chosen on purpose a rather complicated solution whose volume is not as intuitive as the AdS-Reissner-Nordstrom, and we will see how the formalism clarifies what the volume is. The thermodynamics of R-charged black holes has been studied in [33]. In (3+1) dimension, the action is given by:

$$L = \left(\frac{R}{16\pi} - \frac{1}{8\pi} \sum_{I=1}^4 e^{\vec{a}_I \cdot \vec{\phi}} F_{(I)}^2 - \frac{1}{32\pi} \sum_{i=1}^3 ((\partial\phi_i)^2 - \mathcal{V}(\phi_i)) \right) \varepsilon \quad (2.12)$$

with

$$a_1 = (1, 1, 1), a_2 = (1, -1, -1), a_3 = (-1, 1, -1), a_4 = (-1, -1, 1)$$

and

$$\mathcal{V}(\phi_i) = -\frac{g^2}{4\pi} \sum_i \cosh \phi_i \quad (2.13)$$

The metric together with the matter fields are given by:

$$ds^2 = -\prod_{I=1}^4 H_I^{-1/2} f dt^2 + \prod_{I=1}^4 H_I^{1/2} \left(\frac{dr^2}{f} + r^2 d\Omega^2 \right) \quad (2.14)$$

$$A^I = \frac{\sqrt{q_I(q_I + 2m)}}{r + q_I} dt \quad (2.15)$$

$$e^{-\frac{1}{2}\vec{a}_I \cdot \vec{\phi}} = \frac{\prod_{J=1}^4 H_J^{1/4}}{H_I} \quad (2.16)$$

$$f = 1 - \frac{2m}{r} + g^2 r^2 \prod_J H_J \quad (2.17)$$

$$H_J = 1 + \frac{q_J}{r} \quad (2.18)$$

The thermodynamical quantities are:

$$M = m + \frac{1}{4} \sum_{I=1}^4 q_I \quad (2.19)$$

$$Q_I = \frac{1}{2} \sqrt{q_I(q_I + 2m)} \quad (2.20)$$

$$S = \pi \prod_{I=1}^4 \sqrt{r_+ + q_I} \quad (2.21)$$

$$T = \frac{f'(r_+)}{4\pi} \prod_{I=1}^4 H_I^{-1/2}(r_+) \quad (2.22)$$

$$\Phi^I = \frac{\sqrt{q_I(q_I + 2m)}}{2(r_+ + q_I)} \quad (2.23)$$

In the extended phase space, the pressure is the cosmological constant, which is also the bottom of the scalar potential:

$$P = \frac{3}{8\pi} g^2 \quad (2.24)$$

As mentioned in the introduction, the ADM mass is now reinterpreted as the enthalpy and the black hole's volume can be computed using the familiar thermodynamic formula:

$$V := \left(\frac{\partial M}{\partial P} \right)_{Q_I, S} = \frac{\pi}{3} r_+^3 \prod_{J=1}^4 H_J(r_+) \sum_{K=1}^4 H_K(r_+)^{-1} \quad (2.25)$$

In particular, the AdS-RN black hole is a special case when all 4 charges coincide $q_1 = q_2 = q_3 = q$. In this case, the above reduces to:

$$V = \frac{4}{3} \pi (q + r_+)^3 \quad (2.26)$$

Also, the radial coordinate has to be redefined by $r \rightarrow r - q$ in order to recover the usual Schwarzschild-like form of the AdS-RN metric. We then recognize the volume of the AdS-RN black hole in the form of equation (1.2).

The paper [33] asks the interesting question of what integral over the black hole interior would give rise to the volume (2.25). To answer this question, one can recast the above in the form:

$$V = \int_{S^2} \int_{r_0}^{r_+} V'(r) dr d\Omega_2^2 \quad (2.27)$$

where $V(r)$ is the function defined in equation (2.25) (with r_+ relabeled to r), and r_0 is taken to be the largest root of the equation $V(r) = 0$. We then find that r_0 is the largest root of a cubic polynomial:

$$4r_0^3 + 3r_0^2 \sum_I q_I + 2r_0 \sum_{i < j} q_I q_J + \sum_{I < J < K} q_I q_J q_K = 0 \quad (2.28)$$

As for the integral $V'(r)$, it was pointed out in [33] that it is essentially the scalar potential:

$$V = -\frac{8\pi}{3g^2} \int_{S^2} \int_{r_0}^{r_+} \mathcal{V} \sqrt{-g} dr d\Omega_2^2 \quad (2.29)$$

Two aspects of this formula are remarkable: first, the fact that the integrand admits a clean interpretation in terms of the scalar potential; and secondly, the integral does not run over the whole of the black hole's interior. As one can generally expect the volume to have something to do with the scalar potential, the second aspect is perhaps a bit more mysterious than the first one. We now proceed to apply the extended Iyer-Wald formalism to compute the volume, and we will see how the formalism sheds light on the two mysterious aspects as described above. The symplectic potential current and Noether charge for this theory are given by:

$$\Theta^a = \nabla_b (g^{ad} g^{bc} \delta g_{cd} - g^{ab} g^{cd} \delta g_{cd}) - \sum_{i=1}^3 \nabla^a \phi_i \delta \phi_i - 8 \sum_I e^{\vec{a}_I \cdot \vec{\phi}} F_{(I)}^{ab} (\partial_b (\xi^c A_{(I)c}) + \xi^c F_{(I)cb}) \quad (2.30)$$

$$Q^{ab} = -\frac{1}{16\pi} \nabla^{[b} \xi^{a]} + \frac{1}{4\pi} \sum_I e^{\vec{a}_I \cdot \vec{\phi}} F_{(I)}^{ab} \xi^c A_{(I)c} \quad (2.31)$$

Here we only give the on-shell form of these expressions. Next, let us perturb the coupling g^2 . From equations (2.15) and (2.16), it is clear that the matter fields are unaffected, and only the gravity part contributes to δQ and Θ . After some algebra, we find that the only nonzero component of χ is:

$$\chi^{rt} = -\frac{1}{16\pi} \left(\frac{r}{2} \right) \left(2\sqrt{H_1 H_2 H_3 H_4} + r \partial_r \sqrt{H_1 H_2 H_3 H_4} \right) \delta g^2 \quad (2.32)$$

The integral of χ over infinity diverges. If we regularize by a radial cutoff $r_c \gg r_+$, we find:

$$\int_{\infty} \chi = \left[-\frac{r_c^3}{2} - \frac{3r_c^2}{8} \sum_I q_I - \frac{r_c}{4} \sum_{I < J} q_I q_J - \frac{1}{8} \sum_{I < J < K} q_I q_J q_K \right] \delta g^2 \quad (2.33)$$

Next, let us focus on the $\delta \Lambda$ term in the extended first law. By differentiating the Lagrangian with respect to the coupling g^2 , we have:

$$\delta g^2 \int \frac{\partial \mathcal{L}}{\partial g^2} \xi \cdot \varepsilon = -\frac{\delta g^2}{g^2} \int_{S^2} \int_{r_+}^{\infty} \mathcal{V} \sqrt{-g} dr d\Omega_2^2 \quad (2.34)$$

Notice that we have an integral of the scalar potential on the right-hand side! We emphasize here that the extended Iyer-Wald formalism makes this fact manifest, in contrast with the

approach described in equation (2.27). As usual, the upper limit of integration above will diverge and we have to regularize by a radial cutoff r_c . Evaluating the integral, we then find:

$$\delta g^2 \int \frac{\partial \mathcal{L}}{\partial g^2} \xi \cdot \varepsilon = \delta g^2 \left[\frac{r^3}{2} + \frac{3}{8} r^2 \sum_I q_I + \frac{1}{4} r \sum_{I < J} q_I q_J \right]_{r_+}^{r_c} \quad (2.35)$$

If we now compare the divergent terms in (2.33) and (2.35), we then find that they cancel pairwise, and we are left with a finite answer which consists of two parts: (1) the finite term in (2.33) and the horizon term (the lower limit of integration) in (2.35). We then obtain:

$$\begin{aligned} VdP &= \int_{\infty} \chi + \delta g^2 \int \frac{\partial \mathcal{L}}{\partial g^2} \xi \cdot \epsilon \\ &= \delta g^2 \left(\frac{r_+^3}{2} + \frac{3}{8} r_+^2 \sum_I q_I + \frac{r_+}{4} \sum_{I < J} q_I q_J - \frac{1}{8} \sum_{I < J < K} q_I q_J q_K \right) \end{aligned} \quad (2.36)$$

and we recover equation (2.25). Notice in particular, that, from the viewpoint of the extended Iyer-Wald formalism, the lower limit of integration r_0 in (2.29) arises from the finite term in the integral of χ at infinity. Moreover, the Iyer-Wald formalism has taught us that the volume of the black hole is perhaps best thought of as arising from an integral over the *exterior* of the black hole rather than its *interior*². To summarize, the volume arises as the integral of the scalar potential over the whole black hole exterior, but it is regularized by the Iyer-Wald form χ at infinity in a nontrivial way.

3 Volume and Complexity: Schwarzschild-AdS

From the viewpoint of the Iyer-Wald formalism, as we have seen above, the black hole volume arises as an integral over the *exterior* of the black hole. This observation naturally begs the question of whether the thermodynamic volume has something to do with the black hole *interior*. Moreover, it remains unclear as to what the information contained in the volume can teach us about the dual CFT. It is generally pointed out in the literature [5, 9–11, 13, 15, 16, 32] that varying the cosmological constant in the bulk corresponds to varying the rank of $SU(N)$ or the central charge on the field theory side, and that the volume can be thought of as a chemical potential-like quantity corresponding to the degrees of freedom counted by the central charge.

In this section, we bring the two questions above together (the black hole interior and the CFT interpretation) and attempt to answer them through the notion of *complexity* of quantum states. In a series of elegant papers [27, 28], Brown et al. proposed that the complexity of the CFT state is dual to the integral of the bulk action over the so-called Wheeler-DeWitt

²We also note here that there exists an alternative approach in the literature to derive the volume, which is based on the Komar formula, see e.g. [5, 8, 33]. As with our approach, the volume also arises from the Komar formula as a integral over the exterior of the black hole, and the integrand is basically the Killing potential.

(WdW) patch. In particular, this quantity is found to grow linearly with time at late time, and we will see that the thermodynamical volume is a contribution to this growth. Let us start by reviewing the proposal by Brown et al. in some level of details below.

3.1 Review of Brown et al.

The Wheeler-DeWitt patch is a region in the maximally extended black hole spacetime defined with respect to two choices of time, one on each boundary. For simplicity let us first consider the AdS-Schwarzschild black hole in 4 dimensions. We will denote the time on the left boundary as t_L and the time on the right boundary as t_R . From these two points on the boundary (see 1 for a depiction), we draw four null rays, and the WdW patch is the region in the bulk enclosed between rays (and possibly the past and future singularities).

On the CFT side, picking out two times t_L and t_R is equivalent to choosing a quantum state:

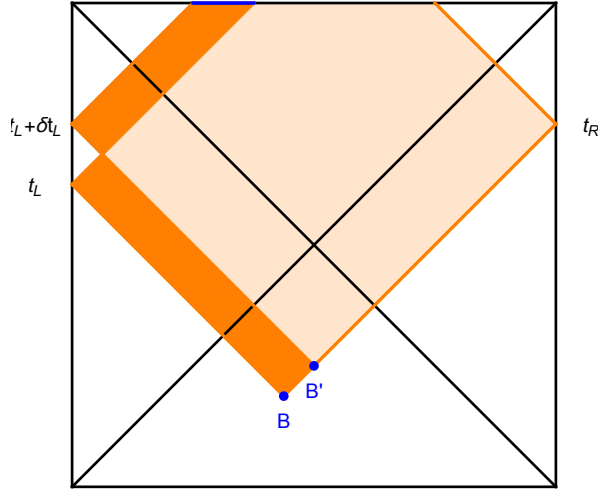


Figure 1. The Wheeler-DeWitt patch of the AdS-Schwarzschild black hole (depicted in orange). When t_L is shifted to $t_L + \delta t_L$, the patch loses a sliver and gains another one (depicted in darker orange). The contributions from the Gibbons-Hawking term are in blue.

$$|\psi(t_L, t_R)\rangle = e^{-i(H_L t_L + H_R t_R)} |TFD\rangle \quad (3.1)$$

where H_L and H_R are the Hamiltonian on the left and right boundaries, respectively, and $|TFD\rangle$ is the thermofield double state:

$$|TFD\rangle = Z^{-1/2} \sum_n e^{-\beta E_n/2} |E_n\rangle_L \otimes |E_n\rangle_R \quad (3.2)$$

The thermofield double state has the properties that it is maximally entangled, and that the reduced density matrix on either side is the usual thermal state. The complexity of a quantum

state is, roughly speaking, the number of quantum gates needed to produce the state from some universally agreed-upon starting point. The statement of Brown et al. is that:

$$\mathcal{C}(|\psi(t_L, t_R)\rangle) = \frac{\mathcal{A}}{\pi\hbar} \quad (3.3)$$

where \mathcal{A} is the bulk action evaluated on the WdW patch. At late t_L , it is then noted that the rate of growth of the complexity approaches the mass of the black hole:

$$\lim_{t_L \rightarrow \infty} \frac{d\mathcal{C}}{dt_L} = \frac{2M}{\pi\hbar} \quad (3.4)$$

This limit is reminiscent of a conjectured upper bound on the rate of computation by Lloyd ([35]), according to which the rate of computation is bounded above by the energy. Comparing with (3.4), and equating the ADM mass with the energy, we find that black holes saturate this conjectured bound at late time.

3.2 Volume as Complexity Growth

In this subsection, we take a closer look at the derivation of (3.4). The method of computation used by Brown et al. was questioned by [34], where the calculation was redone with a more careful treatment of the boundary of the WdW patch. However the conclusion 3.4 remains unchanged. First, since the WdW patch is a region with boundary, the action is the sum of the Einstein-Hilbert action and the Gibbons-Hawking(-York) term:

$$\mathcal{A} = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{|h|} K \quad (3.5)$$

When we shift t_L to $t_L + \delta t_L$, the WdW patch loses a thin rectangle and gains another thin rectangle as described in dark orange in Figure (1). Thus, to compute the rate of change of the action we have to evaluate the action above on the two orange rectangles. Observe that all the sides of these two rectangles are null except at the singularity, and the paper [34] gives a detailed argument that the null boundaries do not contribute to the Gibbons-Hawking term. Also, since the boundary is not smooth at the corners of the rectangles, we have to take into account the contributions localized at these corners (named B and B' in Figure 1). Thus, we see that the Gibbons-Hawking term contributes at the singularity, at B and at B' (all of which are depicted in blue in Figure 1):

$$\delta\mathcal{A} = S_{\mathcal{V}_1} - S_{\mathcal{V}_2} - \frac{1}{8\pi G} \int_S K d\Sigma + \frac{1}{8\pi G} \oint_{B'} a dS - \frac{1}{8\pi G} \oint_B a dS \quad (3.6)$$

Note that \mathcal{V}_1 and \mathcal{V}_2 denote the upper and lower dark orange slivers from figure 1 respectively, and that $a = \ln |k \cdot \bar{k}|$ where k and \bar{k} are the null normals to the corner pieces. Let us consider first the difference between the two rectangles $S_{\mathcal{V}_1} - S_{\mathcal{V}_2}$. Note that the Ricci scalar of the AdS-Schwarzschild solution is a constant:

$$R = \frac{2d}{d-2} \Lambda \quad (3.7)$$

This readily follows from the fact that AdS-Schwarzschild is a vacuum solution of Einstein-Hilbert theory. Thus, if we evaluate the Einstein-Hilbert action on the AdS-Schwarzschild background, we immediately see that we have something proportional to the *spacetime volume*:

$$S_{\mathcal{V}_1} - S_{\mathcal{V}_2} \propto \int_{\mathcal{V}_1} \sqrt{-g} d^4x - \int_{\mathcal{V}_2} \sqrt{-g} d^4x \quad (3.8)$$

Thus, after one evaluates the integrals above, we expect to see something which is schematically the product of a spatial volume and the infinitesimal time interval δt :

$$S_{\mathcal{V}_1} - S_{\mathcal{V}_2} = (\text{some spatial volume}) \delta t \quad (3.9)$$

Let now us do the integral for $S_{\mathcal{V}_1} - S_{\mathcal{V}_2}$ explicitly. When we do this, two remarkable things happen. The is that the part of the upper rectangle which is outside the future horizon always cancels with the part of the lower rectangle which is outside the past horizon, and this happens for any t_L . Thus, whatever quantity comes out to be the spatial volume in equation (3.9) only receives contribution from the black hole interior. The second is that the integral evaluates to:

$$S_{\mathcal{V}_1} - S_{\mathcal{V}_2} = -\frac{r_B^3}{2GL^2} \delta t \quad (3.10)$$

where r_B is the r coordinate of the 2-sphere sitting at B . In the late time limit, we can easily see by inspection of Figure (1) that r_B tends to r_+ . Thus, in the late time limit, the integral above can be interpreted in the language of the extended thermodynamics as:

$$S_{\mathcal{V}_1} - S_{\mathcal{V}_2} = -\frac{r_+^3}{2GL^2} \delta t = -PV \delta t \quad (3.11)$$

Let us pause for a minute to take stock: the calculation above justifies the idea that the thermodynamic volume is the volume of the black hole interior, and in the same time, relates it to the late-time rate of growth of the complexity. Incidentally, an earlier version of the complexity=action proposal postulates that the complexity is equal to the *volume* of a wormhole in the bulk, and is dubbed the complexity=volume duality (or CV duality). From equation (3.11), one can say that the extended thermodynamics gives a new meaning to the phrase “complexity=volume,” namely we might propose that complexity is dual to the spacetime volume of the Wheeler-DeWitt patch. We will discuss this alternate proposal (which we will dub “complexity = volume 2.0”) in more depth a bit later.

3.3 Smarr Relation as Complexity Growth

Let us now press on and evaluate the remaining contributions in (3.6). The algebraic details are found again in ([34]). The new part is that we will recast all the answers in terms of the thermodynamical variables of the black hole. For example, the contribution of the Gibbons-Hawking term at the singularity is essentially the ADM mass:

$$-\frac{1}{8\pi G} \int_S K d\Sigma = \frac{3}{2} M \delta t \quad (3.12)$$

As for the contribution at the two corners B and B' , one finds:

$$\frac{1}{8\pi G} \left[\oint_{B'} adS - \oint_B adS \right] = \frac{1}{4G} \left[r^2 \frac{df}{dr} + 2rf \log \left(\frac{-f}{K} \right) \right]_{r=r_B} \delta t \quad (3.13)$$

where K is a constant. In the late time limit, where $r_B \rightarrow r_+$, the second term above vanishes and:

$$\frac{1}{8\pi G} \left[\oint_{B'} adS - \oint_B adS \right] = TS \delta t \quad (3.14)$$

Putting everything together, we find the time derivative of the action to be:

$$\frac{d\mathcal{A}}{dt} = \frac{3}{2}M + TS - PV \quad (3.15)$$

Next, recall the Smarr relation for AdS-Schwarzschild in 4 dimensions:

$$M = 2TS - 2PV \quad (3.16)$$

Using the Smarr relation above, the time derivative of the action simplifies to:

$$\frac{d\mathcal{A}}{dt} = 2M \quad (3.17)$$

If we now turn the logics around, we can reinterpret the following slight rewriting of the Smarr relation:

$$2M = \frac{3}{2}M + TS - PV \quad (3.18)$$

as a way to keep track of the different contributions to the complexity growth: the left-hand side corresponds to the total growth, the term with M on the right-hand side is the contribution from the singularity of the WdW patch, the term with TS is the corner contributions which end up on the horizon at late time, and finally the term with PV is the contribution from the black hole interior away from the singularity.

3.4 Complexity = Volume 2.0

As seen above, at least in the case of AdS-Schwarzschild, the late time rate of change of the bulk action in the Wheeler-DeWitt patch gives the product PV . We could thus propose the late time rate of change of just the bulk part of the action, divided of course by the pressure, as the correct geometric meaning of thermodynamic volume. This, however, seems a bit unnatural. Note though that for vacuum solutions to Einstein-Hilbert gravity, this bulk action is proportional to the spacetime volume, and the constant of proportionality becomes exactly P . This suggests a more natural identification of the thermodynamic volume as the late time rate of change of the *spacetime volume* of the Wheeler-DeWitt patch. Going further in this direction, one could further propose that rather than complexity = action, we could identify

$$\mathcal{C} \propto \frac{\text{Vol(WdW)}}{\hbar GL^2} \quad (3.19)$$

where $\text{Vol}(\text{WdW})$ is the spacetime volume of the Wheeler-DeWitt patch. We will refer to this idea as “Complexity = Volume 2.0”³, since the name “complexity = volume” has already been used for an earlier version of “complexity = action”. In the late time limit, equation (3.19) yields the thermodynamic volume (multiplied by the pressure):

$$\dot{C} \sim PV \quad (3.20)$$

We will discuss further this proposal in the discussion section 5.

4 Volume and Complexity: Conserved Charges

Given the clean connection between the Wheeler-DeWitt patch of the Schwarzschild-AdS black hole with its thermodynamic volume, it is natural to ask whether we can also establish similar connections for other, more complicated, black hole solutions. In this subsection, we explore this question in the context of black holes with a conserved charge, i.e. the electrically charged and rotating black holes. On the field theory side, it is pointed out in [28] that the existence of conserved charges puts constraints on the system and implies that the rate of growth of the complexity at late time is slower than in the case without charges. In the case of an electrically charged black hole, it is conjectured that this growth obeys the bound:

$$\frac{d\mathcal{A}}{dt} \leq 2[(M - \mu Q) - (M - \mu Q)_{\text{gs}}] \quad (4.1)$$

where the subscript “gs” stands for the ground state, which can be either pure AdS or an extremely charged black hole⁴. Similarly, in the case of a rotating hole, the complexity growth at late time was conjectured to satisfy:

$$\frac{d\mathcal{A}}{dt} \leq 2[(M - \Omega J) - (M - \Omega J)_{\text{gs}}] \quad (4.2)$$

On the gravity side, for both charged and rotating black holes, the Penrose diagram is qualitatively the same. In Figure 2, we depict their Penrose diagram together with the Wheeler-DeWitt patch.

It is pointed out in [28] that the conjectured bound (4.1) was found to be *violated* by large AdS-RN black holes when the action of the WdW patch is explicitly computed. Moreover, the discrepancy is rather bad: the late-time rate of growth of the action can be larger than the one predicted by (4.1) by a factor which can be arbitrarily large. To explain this discrepancy, [28] pointed out that the large AdS-RN black hole cannot be dual to a UV-complete theory. However, it was then subsequently argued in [36] that the bound is violated even for small black holes. This clearly indicates that complexity-action duality in its present form

³After the initial version of this paper was released, Adam Brown made available to us a set of his notes in which he considers this very possibility, for which we would like to thank him.

⁴It is suggested in [27] that, at least in theories with light charged particles, the true ground state in cases with large chemical potential should be given by a charged sphere and not the naïve extremal AdS-RN solution.

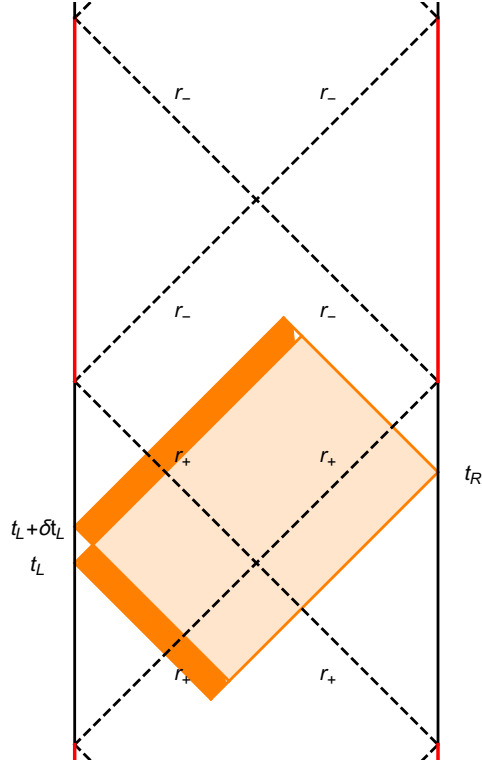


Figure 2. The Penrose diagram of a charged and/or rotating black hole and a Wheeler-DeWitt patch (depicted in orange). When t_L is shifted to $t_L + \delta t_L$, the patch loses a sliver and gains another one (depicted in darker orange). The singularity is in red, and the horizons are dashed.

is not very well understood, and might need to be reformulated. It is our hope that the notion of thermodynamic volume will prove useful toward a more satisfactory formulation of CA-duality. In particular, the complexity = action 2.0 proposal would seem to avoid this problem.

4.1 Charged Black Hole

Let us start with the charged black holes, starting with the simpler case of the charged BTZ. The solution is given by:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 \quad (4.3)$$

$$f(r) = -2m + \frac{r^2}{L^2} - \frac{q^2}{2} \log\left(\frac{r}{L}\right) \quad (4.4)$$

$$A = -q \log\left(\frac{r}{L}\right) dt \quad (4.5)$$

Already we encounter a complication compared to the AdS-Schwarzschild case: with the inclusion of a matter Lagrangian, in general the on-shell action is no longer proportional to

the spacetime volume as in the AdS-Schwarzschild case, and a connection between action and volume as in (3.9) seems *a priori* not so clear. The time derivative of the action on the WdW patch has been computed in [36], and the result is:

$$\frac{d\mathcal{A}}{dt_L} = -\frac{2}{L^2}(r_+^2 - r_-^2) + 2\pi Q^2 \log\left(\frac{r_+}{r_-}\right) \quad (4.6)$$

Note the appearance of the *inner* horizon r_- . This comes from the fact that, in the late time limit, the upper orange rectangle in Figure 2 approaches the inner horizon. On the other hand, the thermodynamic volume has been computed in [8], and the result is:

$$V = \pi r_+^2 - \frac{\pi}{4} Q^2 L^2 \quad (4.7)$$

The volume consists of two terms: the first term is the “naïve” geometric area of a disk with radius r_+ , and the second term is proportional to the charge Q . Of course, we want to recast the growth rate (4.6) in the language of the extended thermodynamics, but the volume above only involves the outer horizon and not the inner horizon. The appropriate thing to do seems to define a second volume associated with the inner horizon. It should be given by the same algebraic expression as the volume above, except for the replacement $r_+ \rightarrow r_-$:

$$V_- = \pi r_-^2 - \frac{\pi}{4} Q^2 L^2 \quad (4.8)$$

To see that the equation above makes sense, notice that when we differentiate the mass with respect to the pressure (at fixed entropy, etc.) to derive the volume, there is nothing specific to the outer horizon. Moreover, from the viewpoint of the Iyer-Wald formalism, equation (4.8) should come out naturally: in the innermost region $0 < r < r_-$, the vector field ∂_t is still timelike and bifurcate, just as for the outermost region $r > r_+$. Therefore, equation (4.8) should come out of a computation à la Iyer-Wald.

Coming back to equation (4.6), we can then say that the first term on the right-hand side of this equation can be interpreted as a difference of the two volumes, one for each horizon:

$$-\frac{2}{L^2}(r_+^2 - r_-^2) \sim P(V_+ - V_-) \quad (4.9)$$

where we have renamed the usual thermodynamic volume V_+ to distinguish it from the other one. As for the second term on the right-hand side of (4.6), it can be naturally recast in terms of the chemical potentials (one for each horizon):

$$2\pi Q^2 \log\left(\frac{r_+}{r_-}\right) \sim Q(\Phi_+ - \Phi_-) \quad (4.10)$$

Thus, despite the fact that we are dealing with a non-vacuum solution, the late-time growth of the action in the case of the charged BTZ black hole can still be interpreted in the language of the extended black hole thermodynamics, just like for the AdS-Schwarzschild case. The major difference is that the interpretation now involves the volume and the chemical potential,

and not the volume alone. Quite curiously, the Q -dependent term in the volumes V_+ and V_- drops out of the difference $V_+ - V_-$.

Next, we move on to discuss the AdS-RN black hole in $n + 2$ dimensions. The solution is:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_n^2 \quad (4.11)$$

$$f(r) = 1 - \frac{\omega^{n-1}}{r^{n-1}} + \frac{q^2}{r^{2(n-1)}} + \frac{r^2}{L^2} \quad (4.12)$$

$$A = \sqrt{\frac{n}{2(n-1)}} \left(\frac{q}{r_+^{n-1}} - \frac{q}{r_-^{n-1}} \right) dt \quad (4.13)$$

As mentioned in the introduction, the thermodynamic volume is well-known to be the geometric volume of a ball:

$$V = \frac{\text{Vol}(S_n)}{n+1} r_+^{n+1} \quad (4.14)$$

If we compare with the analogous expression (4.7) for the charged BTZ black hole, note that the volume here is really just the naïve geometrical volume, and there is no second term proportional to Q . The time derivative (at late time) of the action has been computed in [34], and was found to be:

$$\frac{d\mathcal{A}}{dt_L} = -2\text{Vol}(S_n) \left(\frac{r_+^{n+1}}{L^2} - \frac{r_-^{n+1}}{L^2} - \frac{q^2}{r_-^{n-1}} + \frac{q^2}{r_+^{n-1}} \right) \quad (4.15)$$

In perfect analogy with the charged BTZ case, the first two terms of the right-hand side can be reinterpreted as the difference between two volumes $P(V_+ - V_-)$, with

$$V = \frac{\text{Vol}(S_n)}{n+1} r_-^{n+1} \quad (4.16)$$

and the last two terms in (4.15) can be recast as a difference of two chemical potentials. Once again, like in the charged BTZ case, the growth of the action can be neatly recast in the language of the extended thermodynamics despite the fact that the solution is non-vacuum.

4.2 Rotating Black Hole

Next, we move on to discuss rotating holes, starting with the simple case of the rotating BTZ. The metric reads:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi - \frac{J}{2r^2} dt \right)^2 \quad (4.17)$$

The late-time growth of the action was computed in [28]:

$$\frac{d\mathcal{A}}{dt_L} = -\frac{2}{L^2} (r_+^2 - r_-^2) \quad (4.18)$$

and the thermodynamic volume was computed in [8]:

$$V = \pi r_+^2 \quad (4.19)$$

In perfect analogy with the charged BTZ case, we will refer to this volume as V_+ and define a second volume associated to the inner horizon:

$$V_- = \pi r_-^2 \quad (4.20)$$

The late-time growth of the action can then be written as:

$$\frac{d\mathcal{A}}{dt_L} \sim P(V_+ - V_-) \quad (4.21)$$

Next, we move on to discuss the case of rotating black hole (Kerr-AdS). This case is substantially richer and more interesting than the previous one, as the solution is no longer spherically symmetric and the analysis of the thermodynamics is somewhat different depending on whether the spacetime dimension is odd or even (see [33]). For simplicity, we will focus on the 4-dimensional case. The solution is given by:

$$ds^2 = -\frac{(1 + g^2 r^2)\Delta_\theta}{1 - a^2 g^2} dt^2 + \frac{(r^2 + a^2) \sin^2 \theta}{1 - a^2 g^2} d\phi^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{2mr}{\rho^2(1 - a^2 g^2)^2} (\Delta_\theta dt - a \sin^2 \theta d\phi)^2 \quad (4.22)$$

$$\Delta_r = (r^2 + a^2)(1 + g^2 r^2) - 2mr \quad (4.23)$$

$$\Delta_\theta = 1 - a^2 g^2 \cos^2 \theta \quad (4.24)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad (4.25)$$

Here $a = J/M$ is the ratio of the angular momentum to the mass. The late-time growth of the action was computed in [36]:

$$\frac{d\mathcal{A}}{dt_L} = -\frac{1}{2G(L^2 - a^2)} (r_+^3 + a^2 r_+ - r_-^3 - a^2 r_-) \quad (4.26)$$

As for the volume, we have two different notions of volume depending on whether we do the analysis from a non-rotating or rotating frame at infinity. Following [33], we refer to the volume in the non-rotating frame as the thermodynamic volume and the one in the rotating frame as the geometric volume. The latter admits a geometrical interpretation ⁵:

$$V = \frac{1}{3} r_+ A \quad (4.27)$$

where A is the area of the horizon:

$$A = 4\pi \left(\frac{r_+^2 + a^2}{1 - a^2/L^2} \right) \quad (4.28)$$

⁵We also note here that the thermodynamic quantities derived in the rotating frame obey the Smarr relation [33] but not the first law. On the other hand, the thermodynamic quantities derived in the non-rotating frame at infinity do obey a first law (in addition to a Smarr relation) and can be derived from the Iyer-Wald formalism.

Putting the two equations above together, we have:

$$V = \frac{4}{3}\pi r_+ \left(\frac{r_+^2 + a^2}{1 - a^2/L^2} \right) \quad (4.29)$$

Proceeding similarly to the previous cases, we will refer to this volume as V_+ and define a second volume V_- associated to the inner horizon. As in the previous case, one can define the volume V_- associated to the inner horizon by the replacement $r_+ \rightarrow r_-$ in V_+ :

$$V_- = \frac{4}{3}\pi r_- \left(\frac{r_-^2 + a^2}{1 - a^2/L^2} \right) \quad (4.30)$$

Comparing equations (4.26), (4.29) and (4.29) we finally conclude that, as in the previous cases:

$$\frac{d\mathcal{A}}{dt_L} \sim P(V_+ - V_-) \quad (4.31)$$

While the connection between the geometric volume and the rate of growth of the WdW patch in this case can seem more mysterious than the previous cases, it is important to recall that the Kerr solution is still a vacuum solution. Therefore, as in the AdS-Schwarzschild case, the Ricci scalar is constant and the on-shell action reduces to the spacetime volume, and one can expect a connection between the action and some notion of spatial volume. Nothing guarantees, a priori, that this notion of spatial volume comes out to be the thermodynamic volume. The fact that it does work out that way should be impressive evidence that the thermodynamic/geometric volume of black holes is a relevant quantity to talk about the complexity.

5 Discussion and conclusion

What is the nature of the black hole interior? According to black hole complementarity [37, 38], the black hole interior does not really exist, being just the same quantum system as the black hole exterior but viewed in a different basis. In this view, the classical spacetime region inside a black hole solution should be thought of as an example of emergent space. A quantity which purports to be a measure of the black hole interior's size is the thermodynamic volume, which appears when the cosmological constant is thought of as a pressure variable.

In this paper, we analyzed the thermodynamic volume of black holes from two different angles. Firstly, we used a minor extension of the Iyer-Wald formalism to compute the volume. Our method provides a systematic way to write the volume as an integral over spacetime, in contrast with the usual method of differentiating the mass with respect to the pressure (at fixed entropy). We then apply this method to study the volume of a particular case: the R-charged black hole. In this case, we have learned that the Iyer-Wald formalism explains why the volume is an integral of the scalar potential, a fact which is otherwise a bit mysterious. Moreover, the Iyer-Wald formalism gives us an integral over the whole exterior of the black hole. This integral diverges but another term in the Iyer-Wald formalism (the integral of

χ at infinity) regularizes the integral, thereby introducing a finite contribution. This finite contribution explains the lower limit of integration r_0 when the volume is written as an integral over the black hole interior.

In the second part of this paper, we also provided evidence to support adding the thermodynamic volume to the long and growing list of gravity quantities which admit interpretations in terms of quantum information theory. Specifically, we showed in a broad class of solutions that the time derivative of the bulk action, when evaluated on the Wheeler-DeWitt patch, can be understood in the terms of the extended black hole thermodynamics. We point out that an underlying reason for this is the fact the Iyer-Wald formalism, as previously argued, allows for a derivation of the thermodynamic volume from the action. Also, the Smarr relation itself was shown to come out of the time-derivative of the WdW patch.

Before closing, we would like to discuss a bit further the relation between complexity and thermodynamics. A commonly encountered statement found in the literature is the fact that the time derivative of the complexity should be proportional to the product of the temperature and entropy (see for example [39]):

$$\dot{\mathcal{C}} \sim TS \tag{5.1}$$

The relation above is quite appealing from the viewpoint of quantum circuits: one would expect the complexity to be an extensive quantity, and therefore should scale with the number of qubits. The entropy S is a natural measure of this number. One could naïvely try the mass M of the black hole instead of the combination TS . In the case of extremal black holes, however, this does not work since it would predict a nonzero rate of complexity growth, which would contradict the fact that it is in the ground state and therefore the complexity is not expected to grow. The combination TS vanishes for extremal black holes (since T does) and therefore works well.

Coming back to our proposal of “complexity = volume 2.0”, according to this proposal equation (5.1) above is to be replaced by:

$$\dot{\mathcal{C}} \sim PV \tag{5.2}$$

where, by V , we mean the thermodynamic volume in the case of a single horizon, and the difference $V_+ - V_-$ in the case of a double horizon. We note that, in the case of AdS-RN, “complexity=volume 2.0” seemingly has a number of advantages over the proposal of Brown et al. To start with, it seems to fix the problem with complexity bound violation found in [27]. This is simply because as the charge on the black hole is increased from zero; the outer horizon shrinks, and the inner horizon grows from zero so that the quantity $V_+ - V_-$ shrinks monotonically vanishing in the extremal case. That the volume $V_+ - V_-$ of an extremal black hole vanishes in the extremal case also bodes well, as we expect that the complexity of a ground state will not grow⁶. We believe all of this is true in the Kerr-AdS case as well, though it is harder to see analytically. We have however checked it numerically and in the

⁶We also note here that while the quantity $V_+ - V_-$ tends to zero in the extremal limit, the actual amount

cases we have checked the above statements still apply. Plots for several such numeric checks are given below.

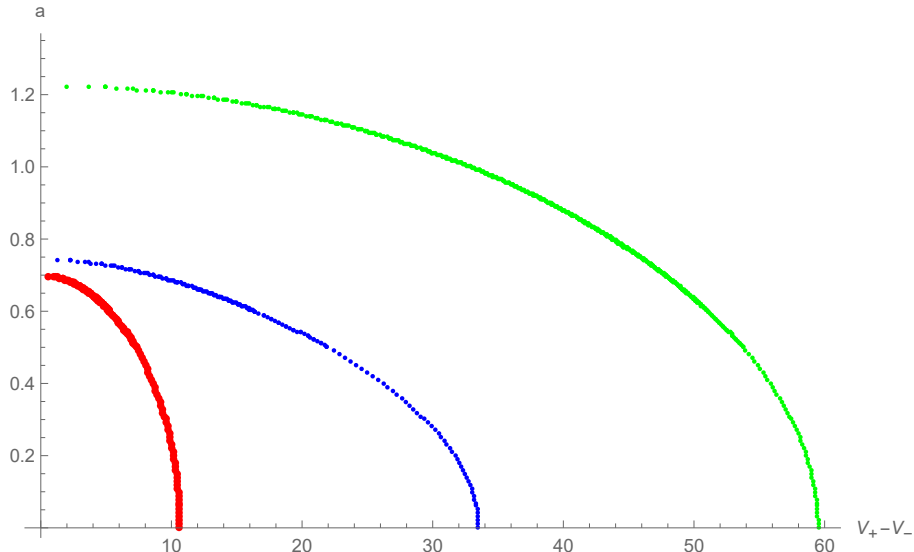


Figure 3. Given M and L , we vary the angular momentum to mass ratio a and for each value solve numerically for $V = V_+ - V_-$. Notice that $a = 0$, which reduces to the Schwarzschild case, has the maximal V . As we approach extremality, which here occurs as the plots flatten out on the left (In flat space extremality occurs for $a = 1$, but this is modified by the AdS length dependence of the metric), V tends towards zero. In the plot green is for $M = 5, L = 1$, blue is for $M = 2, L = 3$, and red is for $M = 1, L = 2$.

Notice in particular that as a increases from 0, $V_+ - V_-$ decreases monotonically. These results are expected, as the extra conservation law puts additional constraints on the system, and we therefore generically expect the complexity to increase at a lower rate.

Finally, we believe this study opens up a number of interesting future directions of research. For example, one could wonder if there is an analog of the thermodynamic volume for dynamical situations such as shock wave geometries or AdS-Vaidya. It would be very interesting to adapt the Iyer-Wald formalism to compute the volume in such a situation and compare with the WdW patch. As many authors have suggested a connection between tensor networks and holography [40–43], another interesting direction would be to consider how thermodynamic volume arises from tensor networks.

6 Acknowledgement

We would like to thank Adam Brown, Bartłomiej Czech, and Leonard Susskind for the useful comments which they provided on an early draft of this paper. We would further like to thank

of space in between the two horizon does not shrink accordingly. This is a reflection of the fact that the thermodynamic volume is not the actual volume of some region in spacetime.

Kimberly Carmona for assisting in proof-reading the manuscript. This material is based upon work supported by the National Science Foundation under Grant Number PHY-1620610.

References

- [1] J. D. Bekenstein, “Black holes and entropy,” *Phys. Rev. D* **7**, 2333 (1973). doi:10.1103/PhysRevD.7.2333
- [2] J. D. Bekenstein, “Generalized second law of thermodynamics in black hole physics,” *Phys. Rev. D* **9**, 3292 (1974). doi:10.1103/PhysRevD.9.3292
- [3] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* **43**, 199 (1975) Erratum: [*Commun. Math. Phys.* **46**, 206 (1976)]. doi:10.1007/BF02345020
- [4] S. W. Hawking, “Black Holes and Thermodynamics,” *Phys. Rev. D* **13**, 191 (1976). doi:10.1103/PhysRevD.13.191
- [5] D. Kastor, S. Ray and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” *Class. Quant. Grav.* **26**, 195011 (2009) doi:10.1088/0264-9381/26/19/195011 [arXiv:0904.2765 [hep-th]].
- [6] D. Kubiznak and R. B. Mann, “Black hole chemistry,” *Can. J. Phys.* **93**, no. 9, 999 (2015) doi:10.1139/cjp-2014-0465 [arXiv:1404.2126 [gr-qc]].
- [7] B. P. Dolan, “Where is the PdV term in the first law of black hole thermodynamics?” arXiv:1209.1272 [gr-qc].
- [8] A. M. Frassino, R. B. Mann and J. R. Mureika, “Lower-Dimensional Black Hole Chemistry,” *Phys. Rev. D* **92**, no. 12, 124069 (2015) doi:10.1103/PhysRevD.92.124069 [arXiv:1509.05481 [gr-qc]].
- [9] D. Kubiznak, R. B. Mann, and M. Teo, “Black hole chemistry: thermodynamics with Lambda,” arXiv:1608.06147 [hep-th].
- [10] C. V. Johnson, “Holographic Heat Engines,” *Class. Quant. Grav.* **31**, 205002 (2014) doi:10.1088/0264-9381/31/20/205002 [arXiv:1404.5982 [hep-th]].
- [11] E. Caceres, P. H. Nguyen, and J. F. Pedraza, “Holographic entanglement entropy and the extended phase structure of STU black holes,” *JHEP* **1509**, 184 (2015) doi:10.1007/JHEP09(2015)184 [arXiv:1507.06069 [hep-th]].
- [12] A. Karch and B. Robinson, “Holographic Black Hole Chemistry,” *JHEP* **1512**, 073 (2015) doi:10.1007/JHEP12(2015)073 [arXiv:1510.02472 [hep-th]].
- [13] E. Caceres, P. H. Nguyen, and J. F. Pedraza, “Holographic Entanglement Chemistry,” arXiv:1605.00595 [hep-th].
- [14] P. H. Nguyen, “An equal area law for holographic entanglement entropy of the AdS-RN black hole,” *JHEP* **1512**, 139 (2015) doi:10.1007/JHEP12(2015)139 [arXiv:1508.01955 [hep-th]].
- [15] D. Kastor, S. Ray and J. Traschen, “Extended First Law for Entanglement Entropy in Lovelock Gravity,” *Entropy* **18**, no. 6, 212 (2016) doi:10.3390/e18060212 [arXiv:1604.04468 [hep-th]].

- [16] D. Kastor, S. Ray and J. Traschen, “Chemical Potential in the First Law for Holographic Entanglement Entropy,” JHEP **1411**, 120 (2014) doi:10.1007/JHEP11(2014)120 [arXiv:1409.3521 [hep-th]].
- [17] P. Pradhan, “Thermodynamic Products in Extended Phase Space,” Int. J. Mod. Phys. D **26**, 1750010 (2017) doi:10.1142/S0218271817500109 [arXiv:1603.07748 [gr-qc]].
- [18] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, “Gravitation from Entanglement in Holographic CFTs,” JHEP **1403**, 051 (2014) doi:10.1007/JHEP03(2014)051 [arXiv:1312.7856 [hep-th]].
- [19] J. Lin, M. Marcolli, H. Ooguri and B. Stoica, “Locality of Gravitational Systems from Entanglement of Conformal Field Theories,” Phys. Rev. Lett. **114**, 221601 (2015) doi:10.1103/PhysRevLett.114.221601 [arXiv:1412.1879 [hep-th]].
- [20] N. Lashkari and M. Van Raamsdonk, “Canonical Energy is Quantum Fisher Information,” JHEP **1604**, 153 (2016) doi:10.1007/JHEP04(2016)153 [arXiv:1508.00897 [hep-th]].
- [21] N. Lashkari, J. Lin, H. Ooguri, B. Stoica and M. Van Raamsdonk, “Gravitational Positive Energy Theorems from Information Inequalities,” arXiv:1605.01075 [hep-th].
- [22] M. Alishahiha, “Holographic Complexity,” Phys. Rev. D **92**, no. 12, 126009 (2015) doi:10.1103/PhysRevD.92.126009 [arXiv:1509.06614 [hep-th]].
- [23] D. Momeni, M. Faizal, S. Bahamonde and R. Myrzakulov, “Holographic complexity for time-dependent backgrounds,” Phys. Lett. B **762**, 276 (2016) doi:10.1016/j.physletb.2016.09.036 [arXiv:1610.01542 [hep-th]].
- [24] D. Momeni, S. A. H. Mansoori and R. Myrzakulov, “Holographic Complexity in Gauge/String Superconductors,” Phys. Lett. B **756**, 354 (2016) doi:10.1016/j.physletb.2016.03.031 [arXiv:1601.03011 [hep-th]].
- [25] O. Ben-Ami and D. Carmi, arXiv:1609.02514 [hep-th].
- [26] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi and K. Watanabe, “Distance between Quantum States and Gauge-Gravity Duality,” Phys. Rev. Lett. **115**, no. 26, 261602 (2015) doi:10.1103/PhysRevLett.115.261602 [arXiv:1507.07555 [hep-th]].
- [27] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Holographic Complexity Equals Bulk Action?,” Phys. Rev. Lett. **116**, no. 19, 191301 (2016) doi:10.1103/PhysRevLett.116.191301 [arXiv:1509.07876 [hep-th]].
- [28] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Complexity, action, and black holes,” Phys. Rev. D **93**, no. 8, 086006 (2016) doi:10.1103/PhysRevD.93.086006 [arXiv:1512.04993 [hep-th]].
- [29] V. Iyer and R. M. Wald, “Some properties of Noether charge and a proposal for dynamical black hole entropy,” Phys. Rev. D **50**, 846 (1994) doi:10.1103/PhysRevD.50.846 [gr-qc/9403028].
- [30] V. Iyer and R. M. Wald, “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” Phys. Rev. D **52**, 4430 (1995) doi:10.1103/PhysRevD.52.4430 [gr-qc/9503052].
- [31] T. Jacobson, G. Kang and R. C. Myers, “On black hole entropy,” Phys. Rev. D **49**, 6587 (1994) doi:10.1103/PhysRevD.49.6587 [gr-qc/9312023].

- [32] B. P. Dolan, “The compressibility of rotating black holes in D-dimensions,” *Class. Quant. Grav.* **31**, 035022 (2014) doi:10.1088/0264-9381/31/3/035022 [arXiv:1308.5403 [gr-qc]].
- [33] M. Cvetič, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume,” *Phys. Rev. D* **84**, 024037 (2011) doi:10.1103/PhysRevD.84.024037 [arXiv:1012.2888 [hep-th]].
- [34] L. Lehner, R. C. Myers, E. Poisson and R. D. Sorkin, “Gravitational action with null boundaries,” arXiv:1609.00207 [hep-th].
- [35] S. Lloyd, “Ultimate physical limits to computation,” *Nature* **406** (2000), no. 6799 10471054.
- [36] R. G. Cai, S. M. Ruan, S. J. Wang, R. Q. Yang and R. H. Peng, “Complexity Growth for AdS Black Holes,” *JHEP* **1609**, 161 (2016) doi:10.1007/JHEP09(2016)161 [arXiv:1606.08307 [gr-qc]].
- [37] L. Susskind, L. Thorlacius and J. Uglum, “The Stretched horizon and black hole complementarity,” *Phys. Rev. D* **48**, 3743 (1993) doi:10.1103/PhysRevD.48.3743 [hep-th/9306069].
- [38] G. ’t Hooft, “On the Quantum Structure of a Black Hole,” *Nucl. Phys. B* **256**, 727 (1985). doi:10.1016/0550-3213(85)90418-3
- [39] L. Susskind, “Entanglement is not enough,” *Fortsch. Phys.* **64**, 49 (2016) doi:10.1002/prop.201500095 [arXiv:1411.0690 [hep-th]].
- [40] B. Czech, L. Lamprou, S. McCandlish and J. Sully, “Integral Geometry and Holography,” *JHEP* **1510**, 175 (2015) doi:10.1007/JHEP10(2015)175 [arXiv:1505.05515 [hep-th]].
- [41] B. Swingle, *Phys. Rev. D* **86**, 065007 (2012) doi:10.1103/PhysRevD.86.065007 [arXiv:0905.1317 [cond-mat.str-el]].
- [42] B. Czech, P. Hayden, N. Lashkari and B. Swingle, “The Information Theoretic Interpretation of the Length of a Curve,” *JHEP* **1506**, 157 (2015) doi:10.1007/JHEP06(2015)157, 10.1007/jhep06(2015)157 [arXiv:1410.1540 [hep-th]].
- [43] B. Czech, G. Evenbly, L. Lamprou, S. McCandlish, X. Qi, J. Sully, G. Vidal, “A tensor network quotient takes the vacuum to the thermal state,” *Phys. Rev. B* **94**, 085101 doi:https://doi.org/10.1103/PhysRevB.94.085101 [arXiv:1510.07637 [cond-mat.str-el]].